Deterministic Identification For MC Poisson Channel

Mohammad J. Salariseddigh*, Vahid Jamali†, Uzi Pereg‡, Holger Boche*, Christian Deppe*, and Robert Schober¶
* Technical University of Munich † Technical University of Darmstadt ‡ Technion ¶ Friedrich-Alexander-University Erlangen-Nürnberg

***Abstract*—Various applications of molecular communications (MC) are event-triggered, and, as a consequence, the prevalent Shannon capacity may not be the right measure for performance assessment. Thus, in this paper, we motivate and establish the identification capacity as an alternative metric. In particular, we study deterministic identification (DI) for the discrete-time Poisson channel (DTPC), subject to an average and a peak power constraint, which serves as a model for MC systems employing molecule counting receivers. It is established that the number of different messages that can be reliably identified for this channel scales as $2^{(n \log n)R}$, where n and R are the codeword length and coding rate, respectively. Lower and upper bounds on the DI capacity of the DTPC are developed.**

I. Introduction

Molecular communication (MC) is a new paradigm in communication engineering where information is transmitted via signaling molecules [1]. Information-theoretical analysis of MC systems is useful for the characterization of their performance limits, guiding MC system design and assessing the efficiency of practical designs against these performance limits. In particular, one of the basic abstract models for MC systems with molecule counting receivers is the discrete-time Poisson channel (DTPC) model [2]. The DTPC model has been used to study the performance limits of MC systems.

Various applications of MC within the framework of sixth generation wireless networks (6G) [3] are associated with event-triggered systems, where Shannon's message transmission capacity, as considered in [2], may not be the appropriate performance metric. In particular, in event-detection scenarios, where the receiver wishes to decides about the occurrence of a specific event in terms of a reliable Yes / No answer, the so-called identification capacity is the relevant performance measure [4], see [3] for MC scenarios for DI.

While in Shannon's communication paradigm [5], the sender, encodes her message in a manner that the receiver can perform reliable detection, in the identification setting [4], the coding scheme is designed to accomplish a different objective, namely, to determine whether a *particular* message was sent or not. In the DI problem, the codewords are selected via a deterministic function from the messages. DI may be preferred over randomized identification (RI) [4] in complexity-constrained applications of MC systems, where the generation of random codewords is challenging. The DI for discrete memoryless and Gaussian channels is studied in [6], where the codebook size scales exponentially and super-exponentially, i.e., $\sim 2^{nR}$ and $\sim 2^{(n \log n)R}$, respectively.

Salariseddigh (mjss@tum.de), Deppe (christian.deppe@tum.de) and Boche (boche@tum.de) were supported by the 6G-Life project, Grant 16KISK002. Jamali (vahid.jamali@tu-darmstadt.de) was supported by the DFG, Grant JA 3104/1-1. Pereg (uzipereg@technion.ac.il) was supported by the Israel CHE Program for Quantum Sci. Tech., Grant 86636903. Schober (robert.schober@fau.de) was supported by MAMOKO, Grant 16KIS0913.

Fig. 1. End-to-end transmission chain for DI communication in a generic MC system modelled as a DTPC. The transmitter maps message i onto a codeword $\mathbf{c}_i = (c_{i,t})|_{t=1}^{n}$. The receiver is provided with an arbitrary message j, and given the channel output vector $\mathbf{Y} = (Y_t)|_{t=1}^{n}$, it asks whether $j = i$ or not.

In this paper, we investigate the fundamental performance limits of DI in MC systems employing molecule counting receivers, where the received signal follows the Poisson law. *Notations*: We adopt the same notations as [3].

II. System Model and Preliminaries

A. System Model

Let $X \in \mathbb{R}_{\geq 0}$ and $Y \in \mathbb{N}_0$ denote random variables (RVs) modeling the rate of molecule release by the transmitter and the number of molecules observed at the receiver, respectively. For the DTPC, the channel model reads $Y = \text{Pois}\,(\rho X + \lambda)$, where ρX and $\lambda \in \mathbb{R}_{>0}$ represents the mean number of observed molecules due to release by transmitter and interfering molecules, respectively; see [3] for further details. The letter-wise conditional distribution of the DTPC output is given by $W(y|x) = e^{-(\rho x + \lambda)}(\rho x + \lambda)^y / y!$. We assume that different channel uses are orthogonal. Hence, for n channel uses, the transition probability law reads

$$W^n(\mathbf{y}|\mathbf{x}) = \prod_{t=1}^{n} W(y_t|x_t) = \prod_{t=1}^{n} \frac{e^{-(\rho x_t + \lambda)}(\rho x_t + \lambda)^{y_t}}{y_t!} \,,$$

The codewords are subject to peak $P_{\text{max}} > 0$ and average power $P_{\text{avg}} > 0$ constraints, where they restrict the rate of molecule release per channel use and over the entire n channel uses in each codeword, respectively; see Section II-B.

B. DI Coding For The DTPC

Definition 1 (Poisson DI Code). An $\big(L(n,R), n, \lambda_1, \lambda_2\big)$ DI code for a DTPC $\mathcal{W}$ under average and peak power constraints of P_{ave} and P_{max}, respectively, and for integer $L(n,R)$, where n and R are the codeword length and coding rate, respectively, is defined as a system $(\mathcal{U}, \mathscr{D})$ which consists of a codebook $\mathcal{U} = \{\mathbf{u}_i\}_{i \in [\![L]\!]} \subset \mathbb{R}_+^n$, such that $0 \leq u_{i,t} \leq P_{\text{max}}$ and $n^{-1} \sum_{t=1}^{n} u_{i,t} \leq P_{\text{avg}}\,, \forall i \in [\![L]\!]\,, \forall t \in [\![n]\!]$, and a collection of decoding regions $\mathscr{D} = \{\mathcal{D}_i\}_{i \in [\![L]\!]}$ with $\bigcup_{i=1}^{L(n,R)} \mathcal{D}_i \subset \mathbb{N}_0^n$. The error probabilities of $(\mathcal{U}, \mathscr{D})$ reads

$$P_{e,1}(i) = 1 - \sum_{\mathbf{y} \in \mathcal{D}_i} W^n\big(\mathbf{y} \,\big|\, \mathbf{u}_i\big) \,,\; P_{e,2}(i,j) = \sum_{\mathbf{y} \in \mathcal{D}_j} W^n\big(\mathbf{y} \,\big|\, \mathbf{u}_i\big)$$

where $P_{e,1}(i) \leq \lambda_1$ and $P_{e,2}(i,j) \leq \lambda_2\,, \forall i, j \in [\![L]\!]$, such that $i_1 \neq i_2$, and arbitrary $\lambda_1, \lambda_2 > 0$. A rate $R > 0$ is called achievable if for arbitrary $\lambda_1, \lambda_2 > 0$ and sufficiently large n, there exists an $(L(n,R), n, \lambda_1, \lambda_2)$ DI code. The operational DI capacity of the DTPC, $\mathbb{C}_{DI}(\mathcal{W}, L)$, is the supremum of all achievable rates.

III. DI Capacity of the DTPC

Theorem 1 The DI capacity of the DTPC[a] $\mathcal{W}$ subject to average and peak power constraints of the form $n^{-1}\sum_{t=1}^{n} u_{i,t} \leq P_{\text{ave}}$ and $0 \leq u_{i,t} \leq P_{\text{max}}$, respectively, in the super-exponential scale, i.e., $L(n,R) = 2^{(n\log n)R}$, is bounded by

$$\frac{1}{4} \leq \mathbb{C}_{DI}(\mathcal{W}, L) \leq \frac{3}{2}\,.$$

[a] Considering a Poisson channel model in MC is justified when the number of released molecules, for which the probability of observation is small, is large; see [7, Sec. IV] for further details.

Proof. The proof of Theorem 1 consisting of achievability and converse, is given in Sections III-A and III-B, respectively. □

A. Achievability

- **Step 1**: We propose a codebook construction and derive an analytical lower bound on the corresponding codebook size using inequalities for sphere packing density.
- **Step 2**: To prove that this codebook leads to an achievable rate, we propose a decoder and show that the corresponding type I and type II error rates vanished as $n \to \infty$.

Codebook Construction: Let $A = \min(P_{\text{ave}}, P_{\text{max}})$. We use a packing arrangement of non-overlapping hyper spheres of radius $r_0 = \sqrt{n\epsilon_n}$ in a hyper cube with edge length A, with $\epsilon_n = a/n^{\frac{1}{2}(1-b)}$ where $a > 0$ is a fixed constant and $0 < b < 1$ an arbitrarily small. Therefore, by setting $L(n,R) = 2^{(n\log n)R}$ and $r_0 = \sqrt{n\epsilon_n} = \sqrt{a}n^{\frac{1+b}{4}}$, we have

$$R \geq \frac{1}{n\log n}\left[\left(\frac{1-b}{4}\right) n\log n + n\log\left(\frac{A}{e\sqrt{a}}\right) + o(n)\right]\,,$$

and obtain $R \geq \frac{1}{4}$ for $n \to \infty$ and $b \to 0$; see [3] for details.

Encoding: Given message $i \in [\![L]\!]$, transmit $\mathbf{x} = \mathbf{u}_i$.

Decoding: Let $\delta_n = c\rho^2\epsilon_n = c\rho^2 a n^{\frac{1}{2}(b-1)}$ where $0 < b < 1$ is arbitrarily small and $0 < c < 2$ is a constant. To identify whether message $j \in [\![L]\!]$ was sent, the decoder checks whether the channel output $\mathbf{y}$ belongs to the following decoding set: $\mathcal{D}_j = \left\{\mathbf{y} \in \mathbb{N}_0^n \;:\; |D(\mathbf{y};\mathbf{u}_j)| \leq \delta_n\right\}$, where $D(\mathbf{y};\mathbf{u}_j) = n^{-1}\sum_{t=1}^{n}[(y_t - (\rho u_{j,t} + \lambda))^2 - (\rho u_{j,t} + \lambda)]$.

Error Analysis: First, consider type I errors, i.e., the transmitter sends $\mathbf{u}_i$, yet $\mathbf{Y} \notin \mathcal{D}_i\,, \forall i \in [\![L]\!]$. The type I error probability is given by

$$P_{e,1}(i) = \Pr(|D(\mathbf{Y};\mathbf{u}_i)| > \delta_n \,|\, \mathbf{u}_i)\,, \tag{1}$$

where the condition means that $\mathbf{x} = \mathbf{u}_i$ was sent.

Next, consider type II errors, i.e., when $\mathbf{Y} \in \mathcal{D}_j$ while the transmitter sent $\mathbf{u}_i$. Then, for every $i,j \in [\![L]\!]$, where $i \neq j$, the type II error probability is given by

$$P_{e,2}(i,j) = \Pr\left(\left|D(\mathbf{Y};\mathbf{u}_j)\right| \leq \delta_n \,|\, \mathbf{u}_i\right)\,. \tag{2}$$

Applying the Chebyshev's inequality and using standard techniques, we establish the following bounds on (1) and (2):

$$P_{e,1}(i) \leq \mathcal{O}(n^{-b}) \leq \lambda_1\,, \quad P_{e,2}(i,j) \leq \mathcal{O}(n^{-b}) \leq \lambda_2\,,$$

for sufficiently large n and arbitrarily small $e_1, e_2 > 0$; see [3] for derivations. We have thus shown that $\forall \lambda_1, \lambda_2 > 0$ and for sufficiently large n, there exists an $(L(n,R), n, \lambda_1, \lambda_2)$ code.

B. Converse

- **Step 1**: We show in Lemma 1 that for any achievable rate (for which the type I and II error probabilities vanish as $n \to \infty$), the distance between any selected entry of one codeword with any entry of another codeword should be at least larger than a threshold.
- **Step 2**: Employing the Lemma 1, we derive an upper bound on the codebook size of achievable DI codes.

Lemma 1. Suppose that R is an achievable rate for the DTPC. Consider a sequence of $(L(n,R), n, \lambda_1^{(n)}, \lambda_2^{(n)})$ codes $(\mathcal{U}^{(n)}, \mathcal{D}^{(n)})$ such that $\lambda_1^{(n)}$ and $\lambda_2^{(n)}$ tend to zero as $n \to \infty$. Then, given a sufficiently large n, the codebook $\mathcal{U}^{(n)}$ satisfies the following property. For every pair of codewords, $\mathbf{u}_{i_1}$ and $\mathbf{u}_{i_2}$, there exists at least one letter $t \in [\![n]\!]$ such that $\left|1 - (\rho u_{i_2,t} + \lambda)/(\rho u_{i_1,t} + \lambda)\right| > \epsilon'_n\,, \forall i_1, i_2 \in [\![L]\!]$, such that $i_1 \neq i_2$, with $\epsilon'_n = P_{\text{max}}/n^{1+b}$, where $b > 0$ is arbitrarily small constant.

Proof. Fix $\lambda_1, \lambda_2 > 0$. Let $\kappa > 0$ be arbitrarily small constants. Assume to the contrary that there exist two messages i_1 and i_2, where $i_1 \neq i_2$, meeting the error constraints, such that for all $t \in [\![n]\!]$, we have $\left|1 - v_{i_2,t}/v_{i_1,t}\right| \leq \epsilon'_n$, where $v_{i_k,t} = \rho u_{i_k,t} + \lambda,\; k = 1,2$. In order to show contradiction, we exploit the *continuity of the Poisson law* and show that the sum of the two error probabilities, $P_{e,1}(i_1) + P_{e,2}(i_2,i_1)$, converges to one from below, i.e., $\lambda_1 + \lambda_2 \geq P_{e,1}(i_1) + P_{e,2}(i_2,i_1) \geq 1 - 2\kappa$, where $\kappa > 0$ is arbitrary small; see [3] for detailed derivation. Clearly, this is a contradiction and the proof is completed. □

Using Lemma 1, we obtain $\|\mathbf{u}_{i_1} - \mathbf{u}_{i_2}\| \geq \left|u_{i_1,t} - u_{i_2,t}\right| > \epsilon'_n/\rho$. Thus, we can define an arrangement of non-overlapping spheres $\mathcal{S}_{\mathbf{u}_i}(n, \lambda\epsilon'_n)$. Hence, by setting $L(n,R) = 2^{(n\log n)R}$ and $r_0 = \lambda\epsilon'_n/2\rho = \lambda P_{\text{max}}/2\rho n^{1+b}$, we have

$$R \leq \frac{1}{n\log n}\left[\left(\frac{1}{2} + (1+b)\right) n\log n + \mathcal{O}(n)\right]\,,$$

and obtain $R \leq \frac{3}{2}$ for $n \to \infty$ and $b \to 0$; see [3] for details.

References

[1] T. Nakano, A. W. Eckford, and T. Haraguchi, *Molecular Communication*. Cambridge University Press, 2013.

[2] A. Gohari, M. Mirmohseni, and M. Nasiri-Kenari, "Information Theory of Molecular Communication: Directions and Challenges," *IEEE Trans. Mol. Biol. Multi-Scale Commun.*, vol. 2, no. 2, pp. 120–142, 2016.

[3] M. J. Salariseddigh, U. Pereg, H. Boche, C. Deppe, V. Jamali, and R. Schober, "Deterministic Identification For Molecular Communications Over The Poisson Channel," *arXiv:2203.02784*, 2022. [Online]. Available: https://arxiv.org/pdf/2203.02784.pdf

[4] R. Ahlswede and G. Dueck, "Identification Via Channels," *IEEE Trans. Inf. Theory*, vol. 35, no. 1, pp. 15–29, 1989.

[5] C. E. Shannon, "A Mathematical Theory of Communication," *Bell Sys. Tech. J.*, vol. 27, no. 3, pp. 379–423, 1948.

[6] M. J. Salariseddigh, U. Pereg, H. Boche, and C. Deppe, "Deterministic Identification Over Channels With Power Constraints," *IEEE Trans. Inf. Theory*, vol. 68, no. 1, pp. 1–24, 2022.

[7] V. Jamali, A. Ahmadzadeh, W. Wicke, A. Noel, and R. Schober, "Channel Modeling For Diffusive Molecular Communication - A Tutorial Review," *Proc. IEEE*, vol. 107, no. 7, pp. 1256–1301, 2019.